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Online Supplement for "Bending the Heckman Curve: Competing Declines in Learning Capacity and Skill Relevance Over the Life Course"

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This section details the derivation of the canonical result, presented in Eq. 3 of the main text. We start by simplifying the exposition of Eq. 2 of the main text by normalizing the other productivity parameters to unity ($\beta_1 = \beta_2 = 1$). This can be done without loss of generality for the model's primary goal: to compare relative returns across periods. We can then derive a non-recursive form of the skill production function (Eq. 2) by successively substituting θ_{t-1} into θ_t

$$\theta_t = \left(th^{\phi} + \theta_0^{\phi} + \sum_{j=1}^{t} \gamma_j I_j^{\phi} \right)^{\frac{1}{\phi}}, \quad \forall t \in \{1, \ldots, T\}, \tag{16}$$

where the j index is the summation index, introduced to avoid reusing the upper-limit symbol, t. This non-recursive form makes transparent how all past investments determine the skill stock.

Given objective function θ_T, the marginal productivity of a period-t investment is

$$\frac{\partial \theta_T}{\partial I_t} = \gamma_t I_t^{\phi-1} \left(Th^{\phi} + \theta_0^{\phi} + \sum_{j=1}^{T} \gamma_j I_j^{\phi} \right)^{\frac{1}{\phi}-1}. \tag{17}$$

We then compute $R_{l,e}$—the ratio of marginal productivity of a later investment compared to that of an earlier one—by evaluating Eq. 17 at $t = e$ and $t = l$. Holding levels of investment constant ($I_l = I_e$), this ratio reduces to

$$R_{l,e} = \frac{\frac{\partial \theta_T}{\partial I_l}}{\frac{\partial \theta_T}{\partial I_e}} = \frac{\gamma_l}{\gamma_e} \left(\frac{I_l}{I_e} \right)^{\phi-1} = \frac{\gamma_l}{\gamma_e}. \tag{18}$$

Derivation of ratio of marginal products in our model

This section derives the ratio of marginal products $R_{l,e,p}$ presented in Eq. 12 of the main text.

We start by simplifying the exposition of Eq. 8 by normalizing the other productivity parameters to unity ($\beta_1 = \beta_2 = 1$), which is without loss of generality for comparing relative returns across periods. We can then write the non-recursive form of this equation by successively substituting $\theta_{t-1,p}$ into $\theta_{t,p}$:

$$\theta_{t,p} = \left(th^{\phi_p} + \sum_{j=1}^{t} \alpha_p^{t-j} \theta_{j-1,p'}^{\phi_p} + \alpha_p^t \theta_{0,p}^{\phi_p} + \sum_{j=1}^{t} \alpha_p^{t-j} \gamma_{j,p} I_{j,p}^{\phi_p} \right)^{\frac{1}{\phi_p}}, \quad t \in 1, \ldots, T, \tag{19}$$

where, analogous to Eq. 16, the j index is introduced to avoid ambiguity in the summation bounds. In this final expression, each period's investment $I_{j,p}$ is weighted both by period-j learning capacity ($\gamma_{j,p}$) and the extent to which capabilities acquired in period j retain their relevance in period t (α_p^{t-j}). Thus, earlier investments experience exponential decay in their contribution to current skill stock when $\alpha_p < 1$, with the degree of decay determined by the temporal distance $t - j$.

From our production function, Eq. 19, the marginal productivity of period-t investment on one's level of skill profile p in period T is defined by:

$$\frac{\partial \theta_{T,p}}{\partial I_{t,p}} = \alpha_p^{T-t} \gamma_{t,p} I_{t,p}^{\phi_p - 1} \left(Th^{\phi_p} + \sum_{j=1}^{T} \alpha_p^{T-j} \theta_{j-1,p'}^{\phi_p} + \alpha_p^T \theta_{0,p}^{\phi_p} + \sum_{j=1}^{T} \alpha_p^{T-j} \gamma_{j,p} I_{j,p}^{\phi_p} \right)^{\frac{1}{\phi_p} - 1}. \tag{20}$$

For notational convenience, define

$$X = Th^{\phi_p} + \sum_{j=1}^{T} \alpha_p^{T-j} \theta_{j-1,p'}^{\phi_p} + \alpha_p^T \theta_{0,p}^{\phi_p} + \sum_{j=1}^{T} \alpha_p^{T-j} \gamma_{j,p} I_{j,p}^{\phi_p}. \tag{21}$$

The marginal product can then be written as

$$\frac{\partial \theta_{T,p}}{\partial I_{t,p}} = \alpha_p^{T-t} \gamma_{t,p} I_{t,p}^{\phi_p - 1} X^{\frac{1}{\phi_p} - 1}. \tag{22}$$

Evaluating at periods e and l (where $e < l$), we have

$$\frac{\partial \theta_{T,p}}{\partial I_{e,p}} = \alpha_p^{T-e} \gamma_{e,p} I_{e,p}^{\phi_p - 1} X^{\frac{1}{\phi_p} - 1}, \tag{23}$$

$$\frac{\partial \theta_{T,p}}{\partial I_{l,p}} = \alpha_p^{T-l} \gamma_{l,p} I_{l,p}^{\phi_p - 1} X^{\frac{1}{\phi_p} - 1}. \tag{24}$$

Forming the ratio, we have

$$R_{l,e,p} = \frac{\frac{\partial \theta_{T,p}}{\partial I_{l,p}}}{\frac{\partial \theta_{T,p}}{\partial I_{e,p}}} \tag{25}$$

$$= \frac{\alpha_p^{T-l} \gamma_{l,p} I_{l,p}^{\phi_p - 1} X^{\frac{1}{\phi_p} - 1}}{\alpha_p^{T-e} \gamma_{e,p} I_{e,p}^{\phi_p - 1} X^{\frac{1}{\phi_p} - 1}}. \tag{26}$$

Importantly, X is identical in both expressions because it does not depend on whether we're evaluating period e or period l. Then, the terms $X^{\frac{1}{\phi_p}-1}$ cancel out. This simplifies the ratio to

$$R_{l,e,p} = \frac{\alpha_p^{T-l}\gamma_{l,p}I_{l,p}^{\phi_p-1}}{\alpha_p^{T-e}\gamma_{e,p}I_{e,p}^{\phi_p-1}} \tag{27}$$

$$= \alpha_p^{(T-l)-(T-e)}\frac{\gamma_{l,p}}{\gamma_{e,p}}\frac{I_{l,p}^{\phi_p-1}}{I_{e,p}^{\phi_p-1}} \tag{28}$$

$$= \alpha_p^{e-l}\frac{\gamma_{l,p}}{\gamma_{e,p}}\left(\frac{I_{l,p}}{I_{e,p}}\right)^{\phi_p-1}. \tag{29}$$

Holding levels of investment constant ($I_l = I_e$), this ratio reduces to

$$R_{l,e,p} = \frac{\frac{\partial\theta_{T,p}}{\partial I_{l,p}}}{\frac{\partial\theta_{T,p}}{\partial I_{e,p}}} = \alpha_p^{e-l}\frac{\gamma_{l,p}}{\gamma_{e,p}}\left(\frac{I_{l,p}}{I_{e,p}}\right)^{\phi_p-1} = \alpha_p^{e-l}\frac{\gamma_{l,p}}{\gamma_{e,p}}. \tag{30}$$

Substituting the definitions of α_p and $\gamma_{t,p}$ and simplifying, we have:

$$R_{l,e,p} = \left(\frac{1+\delta_{\text{relevance},p}}{1+\delta_{\text{learning},p}}\right)^{n(l-e)}. \tag{31}$$

Online Supplement B Independence from evaluation horizon

Our model evaluates skill stocks at period T, which we set to peak earning years (ages 50–54, $T = 11$). Here we illustrate that, from the derivations presented in the main text, our core result does not directly depend on this choice. Eq. 12 in the main text shows that the ratio of marginal products is

$$R_{l,e,p} = \left(\frac{1 + \delta_{\text{relevance},p}}{1 + \delta_{\text{learning},p}} \right)^{n(l-e)}, \tag{32}$$

which contains no T term. While the evaluation horizon affects absolute skill stock levels, it does not affect our quantity of interest: the relative productivity of investments across periods.

Here we illustrate what happens when we vary the evaluation horizon. Figure S.B.1 computes return trajectories under four different evaluation horizons: $T = 5$ (ages 20–24), $T = 8$ (ages 35–39), $T = 11$ (ages 50–54), and $T = 14$ (ages 65–69), holding $\delta_{\text{learning},p} = 2.5\%$ constant and varying $\delta_{\text{relevance},p}$ across slow (0.5%), moderate (1.5%), and fast (3%) categories. The panels show identical return trajectories across all evaluation horizons, confirming empirically that optimal investment timing is invariant to the choice of T. While longer horizons allow comparison across more periods, the relative productivity between any two periods remains constant regardless of T.

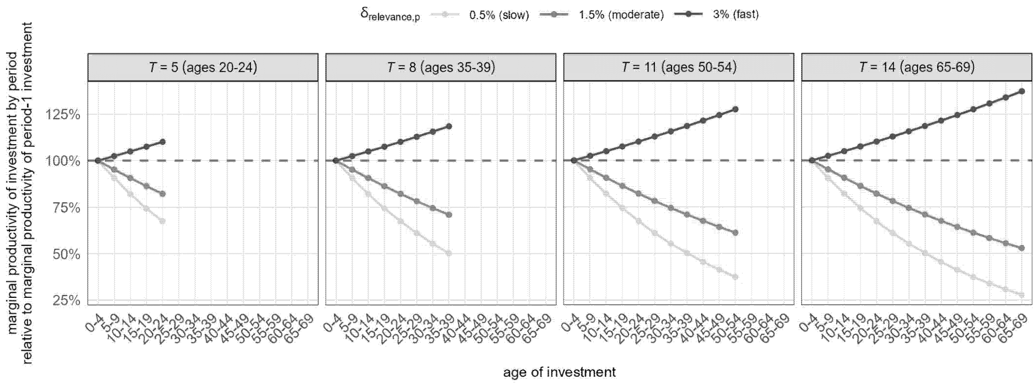

Figure S.B.1: Independence from evaluation horizon. Marginal productivity of investments over the life course relative to early childhood (period 1), evaluated at four different horizons. Return trajectories are identical across evaluation horizons, demonstrating that optimal investment timing depends only on skill-specific decay parameters ($\delta_{\text{learning},p} = 2.5\%$; $\delta_{\text{relevance},p}$ varies across slow, moderate, and fast categories as shown).